Chapter 15 – Sequences and series

Solutions to Exercise 15A

1 a $t_1 = 3$

$t_2 = 3 + 4 = 7$

$t_3 = 7 + 4 = 11$

$t_4 = 11 + 4 = 15$

$t_5 = 15 + 4 = 19$

b $t_1 = 5$

$t_2 = 3 \times 5 + 4 = 19$

$t_3 = 3 \times 19 + 4 = 61$

$t_4 = 3 \times 61 + 4 = 187$

$t_5 = 3 \times 187 + 4 = 565$

c $t_1 = 1$

$t_2 = 5 \times 1 = 5$

$t_3 = 5 \times 5 = 25$

$t_4 = 5 \times 25 = 125$

$t_5 = 5 \times 125 = 625$

d $t_1 = -1$

$t_2 = -1 + 2 = 1$

$t_3 = 1 + 2 = 3$

$t_4 = 3 + 2 = 5$

$t_5 = 5 + 2 = 7$

e $t_1 = 1$

$t_2 = 3$

$t_3 = 2 \times 3 + 1 = 7$

$t_4 = 2 \times 7 + 3 = 17$

$t_5 = 2 \times 17 + 7 = 41$

2 a $t_2 = t_1 + 3$

$t_3 = t_2 + 3$

$\therefore t_n = t_{n-1} + 3,\ t_1 = 3$

b $t_2 = 2t_1$

$t_3 = 2t_2$

$\therefore t_n = 2t_{n-1},\ t_1 = 1$

c $t_2 = -2 \times t_1$

$t_3 = -2 \times t_2$

$\therefore t_n = -2t_{n-1},\ t_1 = 3$

d $t_2 = t_1 + 3$

$t_3 = t_2 + 3$

$\therefore t_n = t_{n-1} + 3,\ t_1 = 4$

e $t_2 = t_1 + 5$

$t_3 = t_2 + 5$

$\therefore t_n = t_{n-1} + 5,\ t_1 = 4$

3 a $t_n = \frac{1}{n}$

$t_1 = \frac{1}{1} = 1$

$t_2 = \frac{1}{2}$

$t_3 = \frac{1}{3}$

$t_4 = \frac{1}{4}$

b $t_n = n^2 + 1$

$t_1 = 1^2 + 1 = 2$

$t_2 = 2^2 + 1 = 5$

$t_3 = 3^2 + 1 = 10$

$t_4 = 4^2 + 1 = 17$

c $t_n = 2n$

$t_1 = 2 \times 1 = 2$

$t_2 = 2 \times 2 = 4$

$t_3 = 2 \times 3 = 6$

$t_4 = 2 \times 4 = 8$

d $t_n = 2^n$

$t_1 = 2^1 = 2$

$t_2 = 2^2 = 4$

$t_3 = 2^3 = 8$

$t_4 = 2^4 = 16$

e $t_n = 3n + 2$

$t_1 = 3 \times 1 + 2 = 5$

$t_2 = 3 \times 2 + 2 = 8$

$t_3 = 3 \times 3 + 2 = 11$

$t_4 = 3 \times 4 + 2 = 14$

f $t_n = (-1)^n n^3$

$t_1 = (-1)^1 \times 1^3 = -1$

$t_2 = (-1)^2 \times 2^3 = 8$

$t_3 = (-1)^3 \times 3^3 = -27$

$t_4 = (-1)^4 \times 4^3 = 64$

g $t_n = 2n + 1$

$t_1 = 2 \times 1 + 1 = 3$

$t_2 = 2 \times 2 + 1 = 5$

$t_3 = 2 \times 3 + 1 = 7$

$t_4 = 2 \times 4 + 1 = 9$

h $t_n = 2 \times 3^{n-1}$

$t_1 = 2 \times 3^0 = 2$

$t_2 = 2 \times 3^1 = 6$

$t_3 = 2 \times 3^2 = 18$

$t_4 = 2 \times 3^3 = 54$

4 **a** $t_n = 3n$

b $t_n = 2^{n-1}$

c $t_n = \dfrac{1}{n^2}$

d $t_n = 3(-2)^{n-1}$

e $t_n = 3n + 1$

f $t_n = 5n - 1$

5 $t_n = 3n + 1$

$t_{n+1} = 3(n + 1) + 1$

$= 3n + 4$

$t_{2n} = 3(2n) + 1$

$= 6n + 1$

6 **a** $t_n = t_{n-1} + 3,\ t_1 = 15$

b $t_1 = 15$

$t_2 = 15 + 3$

$t_3 = (15 + 3) + 3$

$= 15 + 2 \times 3$

$\therefore t_n = 15 + (n - 1) \times 3$

$= 3n + 12$

c $t_{13} = 3 \times 13 + 12$

$= 51$

7 a 4% reduction is equivalent to 96% of the original.
$t_n = 0.96t_{n-1}$
$t_1 = 94.3$

b $t_1 = 94.3$

$t_2 = 0.96 \times 94.3$

$t_3 = 0.96 \times (0.96 \times 94.3)$

$= 0.96^2 \times 94.3$

$\therefore t_n = 94.3 \times 0.96^{n-1}$

c $t_9 = 94.3 \times 0.96^8$

≈ 68.03 seconds

8 a $t_n = 1.8t_{n-1} + 20$
$t_0 = 100$

b $t_1 = 1.8 \times 100 + 20 = 200$

$t_2 = 1.8 \times 200 + 20 = 380$

$t_3 = 1.8 \times 380 + 20 = 704$

$t_4 = 1.8 \times 704 + 20 = 1287$

$t_5 = 1.8 \times 1287 + 20 = 2336$

9 a $t_1 = 2000 \times 1.06$

$= \$2120$

$t_2 = (2120 + 400) \times 1.06$

$= \$2671.20$

$t_3 = (2671.2 + 400) \times 1.06$

$= \$3255.47$

b $t_n = (t_{n-1} + 400) \times 1.06$

$= 1.06(t_{n-1} + 400), \ t_1 = 2120$

c Method will depend on the calculator or spreadsheet used.
$t_{10} = \$8454.02$

10 a 1, 4, 7, 10, 13, 16

b 3, 1, −1, −3, −5, −7

c $\frac{1}{2}$, 1, 2, 4, 8, 16

d 32, 16, 8, 4, 2, 1

11 a 1.1, 1.21, 1.4641, 2.144, 4.595, 21.114

b 27, 18, 12, 8, $\frac{16}{3}$, $\frac{32}{9}$

c −1, 3, 11, 27, 59, 123

d −3, 7, −3, 7, −3, 7

12 a $t_n = 2^{n-1}$

$t_1 = 2^0 = 1$

$t_2 = 2^1 = 2$

$t_3 = 2^2 = 4$

b $u_n = \frac{1}{2}(n^2 - n) + 1$

$u_1 = \frac{1}{2}(1^2 - 1) + 1 = 1$

$u_2 = \frac{1}{2}(2^2 - 2) + 1 = 2$

$u_3 = \frac{1}{2}(3^2 - 3) + 1 = 4$

c The sequences are the same for the first three terms.

$t_1 = u_1$

$t_2 = u_2$

$t_3 = u_3$

d $t_4 = 2^3 = 8$

$u_4 = \frac{1}{2}(4^2 - 4) + 1 = 7$

The sequences are not the same after the first three terms.

13 $S_1 = a \times 1^2 + b \times 1 = a + b$

$S_2 = a \times 2^2 + b \times 2 = 4a + 2b$

$S_3 = a \times 3^2 + b \times 3 = 9a + 3b$

$S_{n+1} - S_n$

$= a(n+1)^2 + b(n+1) - an^2 - bn$

$= a(n^2 + 2n + 1) + bn + b - an^2 - bn$

$= an^2 + 2an + a + b - an^2$

$= 2an + a + b$

14 $t_2 = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2} = 1.5$

$t_3 = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) = \frac{17}{12} \approx 1.4168$

$t_4 = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{17/12}\right) = \frac{577}{408} \approx 1.4142$

Comparing the terms to real numbers between 1 and 1.5, it can be seen that the sequence gives an approximation of $\sqrt{2} = 1.4142$

15

$$
\begin{aligned}
t_3 &= t_2 + t_1 \\
&= 1 + 1 = 2 \\
t_4 &= t_3 + t_2 \\
&= 2 + 1 = 3 \\
t_5 &= t_4 + t_3 \\
&= 3 + 2 = 5 \\
t_{n+2} &= t_{n+1} + t_n \\
\therefore t_{n+1} &= t_n + t_{n-1} \\
\therefore t_{n+2} &= (t_n + t_{n-1}) + t_n \\
&= 2t_n + t_{n-1}
\end{aligned}
$$

Solutions to Exercise 15B

1 $t_n = a + (n - 1)d$

a $t_1 = 0 + (1 - 1) \times 2 = 0$

$t_2 = 0 + (2 - 1) \times 2 = 2$

$t_3 = 0 + (3 - 1) \times 2 = 4$

$t_4 = 0 + (4 - 1) \times 2 = 6$

b $t_1 = -3 + (1 - 1) \times 5 = -3$

$t_2 = -3 + (2 - 1) \times 5 = 2$

$t_3 = -3 + (3 - 1) \times 5 = 7$

$t_4 = -3 + (4 - 1) \times 5 = 12$

c $t_1 = -\sqrt{5} + (1 - 1) \times -\sqrt{5} = -\sqrt{5}$

$t_2 = -\sqrt{5} + (2 - 1) \times -\sqrt{5} = -2\sqrt{5}$

$t_3 = -\sqrt{5} + (3 - 1) \times -\sqrt{5} = -3\sqrt{5}$

$t_4 = -\sqrt{5} + (4 - 1) \times -\sqrt{5} = -4\sqrt{5}$

d $t_1 = 11 + (1 - 1) \times -2 = 11$

$t_2 = 11 + (2 - 1) \times -2 = 9$

$t_3 = 11 + (3 - 1) \times -2 = 7$

$t_4 = 11 + (4 - 1) \times -2 = 5$

2 a $t_{13} = a + 12d$

$= 5 + 12 \times -3 = -31$

b $t_{10} = a + 9d$

$= -12 + 9 \times 4 = 24$

c $t_9 = a + 8d$

$= 25 + 8 \times -2.5 = 5$

d $t_5 = a + 4d$

$= 2\sqrt{3} + 4 \times \sqrt{3}$

$= 6\sqrt{3}$

3 a $a + (1 - 1)d = 3$

$a = 3$

$3 + (2 - 1)d = 7$

$d = 7 - 3 = 4$

$\therefore t_n = 3 + 4(n - 1)$

$= 4n - 1$

b $a + (1 - 1)d = 3$

$a = 3$

$3 + (2 - 1)d = -1$

$d = -1 - 3 = -4$

$\therefore t_n = 3 + -4(n - 1)$

$= 7 - 4n$

c $a + (1 - 1)d = -\frac{1}{2}$

$a = -\frac{1}{2}$

$-\frac{1}{2} + (2 - 1)d = \frac{3}{2}$

$d = \frac{3}{2} - \frac{1}{2} = 2$

$t_n = -\frac{1}{2} + 2(n - 1)$

$= 2n - \frac{5}{2}$

d $a + (1 - 1)d = 5 - \sqrt{5}$

$a = 5 - \sqrt{5}$

$(5 - \sqrt{5}) + (2 - 1)d = 5$

$d = 5 - (5 - \sqrt{5})$

$= \sqrt{5}$

$t_n = (5 - \sqrt{5}) + \sqrt{5}(n - 1)$

$= n\sqrt{5} + 5 - 2\sqrt{5}$

4 **a** $a = 6$ and $d = 4$

$6 + 4(n - 1) = 54$

$4(n - 1) = 48$

$n - 1 = 12$

$n = 13$

b $a = 5$ and $d = -3$

$5 - 3(n - 1) = -16$

$-3(n - 1) = -21$

$n - 1 = 7$

$n = 8$

c $a = 16$ and $d = 16 - 13 = 3$

$16 + 3(n - 1) = -41$

$-3(n - 1) = -57$

$n - 1 = 19$

$n = 20$

d $a = 7$ and $d = 11 - 7 = 4$

$7 + 4(n - 1) = 227$

$4(n - 1) = 220$

$n - 1 = 55$

$n = 56$

5 $t_4 = 7$

$t_{30} = 85$

$a + 3d = 7 \ldots (1)$

$a + 29d = 85 \ldots (2)$

Equation (2) − Equation (1)

$26d = 78$

$d = 3$

$\therefore a = -2$

$t_7 = -2 + 6 \times 3$

$= 16$

6

$a + 2d = 18 \quad \ldots (1)$

$a + 5d = 486 \quad \ldots (2)$

Equation (2) − Equation (1)

$3d = 468$

$d = 156$

$a + 2 \times 156 = 18$

$a + 312 = 18$

$a = -294$

$\therefore t_n = -294 + 156(n - 1)$

$= 156n - 450$

7
$$a + 6d = 0.6 \ldots (1)$$
$$a + 11d = -0.4 \ldots (1)$$
Equation (2) − Equation (1)
$$5d = -1.0$$
$$d = -0.2$$
$$a + 6 \times -0.2 = 0.6$$
$$a - 1.2 = 0.6$$
$$a = 1.8$$
$$\therefore t_{20} = 1.8 + 19 \times -0.2$$
$$= -2$$

8
$$a + 4d = 24 \ldots (1)$$
$$a + 9d = 39 \ldots (2)$$
Equation (2) − Equation (1)
$$5d = 15$$
$$d = 3$$
$$a + 4 \times 3 = 24$$
$$a + 12 = 24$$
$$a = 12$$
$$\therefore t_{15} = 12 + 14 \times 3$$
$$= 54$$

9
$$a + 9d = 31 \ldots (1)$$
$$a + 19d = 61 \ldots (2)$$
Equation (2) − Equation (1)
$$10d = 30$$
$$d = 3$$
Substitute into either equation (1) or (2) to find $a = 4$

$t_6 = 4 + (6 - 1) \times 3 = 19$

10 **a** 672

b 91st week

11 **a** P is the 16th row. $a = 25,\ d = 3$
$$t_{16} = a + 15d$$
$$= 25 + 15 \times 3$$
$$= 70 \text{ seats}$$

b X is the 24th row. $a = 25,\ d = 3$
$$t_{24} = a + 23d$$
$$= 25 + 23 \times 3$$
$$= 94 \text{ seats}$$

c
$$t_n = 25 + 3(n - 1) = 40$$
$$3(n - 1) = 15$$
$$n - 1 = 5$$
$$n = 6$$
Row F

12
$$t_6 = 3 + 5d = 98$$
$$5d = 95$$
$$d = 19$$
$$t_7 = t_6 + 19$$
$$= 117$$

13 $4 + 9d = 30$

$9d = 26$

$d = \frac{29}{9}$

$t_2 = 4 + 1 \times \frac{26}{9} = \frac{62}{9}$

$t_3 = 4 + 2 \times \frac{26}{9} = \frac{88}{9}$

$t_4 = 4 + 3 \times \frac{26}{9} = \frac{38}{3}$

$t_5 = 4 + 4 \times \frac{26}{9} = \frac{140}{9}$

$t_6 = 4 + 5 \times \frac{26}{9} = \frac{166}{9}$

$t_7 = 4 + 6 \times \frac{26}{9} = \frac{64}{3}$

$t_8 = 4 + 7 \times \frac{26}{9} = \frac{218}{9}$

$t_9 = 4 + 8 \times \frac{26}{9} = \frac{244}{9}$

14 $5 + 5d = 15$

$5d = 10$

$d = 2$

$t_2 = 5 + 1 \times 2 = 7$

$t_3 = 5 + 2 \times 2 = 9$

$t_4 = 5 + 3 \times 2 = 11$

$t_5 = 5 + 4 \times 2 = 13$

15 $a + (m - 1)d = 0$

$(m - 1)d = -a$

$d = -\frac{a}{m - 1}$

$t_n = a - \frac{a(n - 1)}{m - 1}$

This could be simplified as follows:

$$t_n = \frac{a(m - 1) - a(n - 1)}{m - 1}$$
$$= \frac{a(m - 1 + n + 1)}{m - 1}$$
$$= \frac{a(m - n)}{m - 1}$$

16 **a** $c = \frac{a + b}{2}$

$= \frac{8 + 15}{2} = 11.5$

b $c = \frac{a + b}{2}$

$= \frac{21 + 79}{2} = 50$

17 $3x - 2 = \frac{5x + 1 + 11}{2}$

$6x - 4 = 5x + 12$

$x = 16$

18 Use the fact that the difference is constant.

$(8a - 13) - (4a - 4) = (4a - 4) - a$

$8a - 13 - 4a + 4 = 4a - 4 - a$

$4a - 9 = 3a - 4$

$a = 5$

19 $t_m = a + (m - a)d = n$

$t_n = a + (n - a)d = m$

Subtract:

$(m - n)d = n - m$

$= -1(m - n)$

$d = \frac{-1(m - n)}{m - n}$

$= -1$

Substitute:

$$a + (m - a) \times -1 = n$$

$$a = m + n - 1$$

$$t_{m+n} = a + (m + n - 1)d$$

$$= n + m - 1 + (m + n - 1) \times -1$$

$$= n + m - 1 - m - n + 1$$

$$= 0$$

20 Use the fact that the difference is constant.

$$a^2 - 2a = 2a - a$$

$$a^2 - 3a = 0$$

$$a(a - 3) = 0$$

$$a = 3 \text{ (since } a \neq 0)$$

21 If a is a prime number, then the nth term is $a + (n - 1)d$ Since a is a natural number there is an n such that $n - 1 = a$. The term $t_{a+1} = a + ad$ which is divisible by a (=$a(d + 1)$ is composite since $d + 1 \geq 2$ and $a \geq 2$). Hence no infinite arithmetic sequence of primes exists.

Solutions to Exercise 15C

1 a $a = 8,\ d = 5,\ n = 12$

$$t_{12} = 8 + 11 \times 5 = 63$$

$$S_{12} = \frac{12}{2}(8 + 63)$$

$$= 6 \times 71$$

$$= 426$$

b $a = -3.5,\ d = 2,\ n = 10$

$$t_{10} = -3.5 + 9 \times 2 = 14.5$$

$$S_{10} = \frac{10}{2}(-3.5 + 14.5)$$

$$= 5 \times 11$$

$$= 55$$

c $a = \frac{1}{\sqrt{2}},\ d = \frac{1}{\sqrt{2}},\ n = 15$

$$t_{15} = \frac{1}{\sqrt{2}} + 14 \times \frac{1}{\sqrt{2}}$$

$$= \frac{15}{\sqrt{2}}$$

$$S_{15} = \frac{15}{2}\left(\frac{1}{\sqrt{2}} + \frac{15}{\sqrt{2}}\right)$$

$$= 60\sqrt{2}$$

d $a = -4,\ d = 5,\ n = 8$

$$t_8 = -4 + 7 \times 5 = 31$$

$$S_8 = \frac{8}{2}(-4 + 31)$$

$$= 108$$

2 $a = 7,\ d = 3,\ n = 7$

$$S_7 = \frac{7}{2}(14 + 6 \times 3)$$

$$= 112$$

3 $a = 5,\ d = 5,\ n = 16$

$$S_{16} = \frac{16}{2}(10 + 15 \times 5)$$

$$= 680$$

4 There will be half of 98 = 49 numbers:

$a = 2,\ d = 2,\ n = 49$

$$S_{49} = \frac{49}{2}(4 + 48 \times 2)$$

$$= 2450$$

5 a 14

b 322

6 a 20

b −280

7 a 12

b 105

8 a 180

b $S_n = \frac{n}{2}(8 + (n-1) \times 4)$

$$= 180$$

$$n(8 + 4n - 4) = 360$$

$$4n^2 + 4n - 360 = 0$$

$$n^2 + n - 90 = 0$$

$$(n - 9)(n + 10) = 0$$

$$n = 9 \text{ as } n > 0$$

So $\{n : S_n = 180\} = \{n : n = 9\}$

9 $S_n = \frac{n}{2}(30 + (n-1) \times -1) = 110$

$n(30 - n + 1) = 220$

$-n^2 + 31n - 220 = 0$

$n^2 - 31n + 220 = 0$

$(n-11)(n-20) = 0$

$n = 11 \text{ or } n = 20$

Reject any value of $n > 15$, as this would involve a negative number of logs in a row. There will be 11 layers.

10 $a = -5,\ d = 4$

$S_m = \frac{m}{2}(-10 + (m-1) \times 4)$

$= 660$

$m(-10 + 4m - 4) = 1320$

$4m^2 - 14m - 1320 = 0$

$(m-20)(4m+66) = 0$

$m = 20 \text{ as } m > 0$

11 $S_n = \frac{n}{2}(a + \ell) \therefore S_n = 0$

12 **a** $a = 6$

$t_{15} = 6 + 14d = 27$

$14d = 21$

$d = 1.5$

$t_8 = 6 + 7 \times 1.5$

$= 16.5 \text{ km}$

b $S_5 = \frac{5}{2}(12 + 4 \times 1.5)$

$= 45 \text{ km}$

c 7 walks

d Total distance:

$S_{15} = \frac{15}{2}(12 + 14 \times 1.5)$

$= 247.5$

Distance missed $= 18 + 19.5 + 21$

$= 58.5 \text{ km}$

(8th day = 16.5 km)

Distance Dora walks $= 247.5 - 58.5$

$= 189 \text{ km}$

13 **a** $a = 30,\ d = 5$

$S_n = \frac{n}{2}(60 + (n-1) \times 5)$

Check:

When $n = 9, S_n = 450$

When $n = 10, S_n = 525$

It will take 10 days

b $a = 50,\ n = 5$

$S_5 = \frac{5}{2}(100 + 4d)$

$= 500$

$100 + 4d = 200$

$d = \frac{200 - 100}{4}$

$= 25$ pages per day

14 **a** Row J $= t_{10}$

$= 50 + 9 \times 4 = 86$

b $S_{26} = \frac{26}{2}(100 + 25 \times 4)$

$= 2600$

c $50 + 54 + 58 + 62 = 224$

d $2600 - 224 = 2376$

e $S_n = \frac{n}{2}(100 + (n-1) \times 4)$

$= 3410$

$$n(100 + 4n - 4) = 6820$$
$$4n^2 + 96n - 6820 = 0$$
$$n^2 + 24n - 1705 = 0$$
$$(n - 31)(n + 55) = 0$$
$$n = 31 \text{ as } n > 0$$

There are 5 extra rows (from 26 to 31).

15 Total members

$$S_{12} = \frac{12}{2}(80 + 11 \times 15)$$
$$= 1470$$
$$\text{Total fees} = 1470 \times \$120$$
$$= \$176\,400$$

16

$$a + d = -12$$
$$6(2a + 11d) = 18$$
$$2a + 11d = 3$$

Substitute $a = -12 - d$:

$$-24 - 2d + 11d = 3$$
$$9d - 24 = 3$$
$$d = 3$$
$$a + 3 = -12$$
$$a = -15$$
$$t_6 = -15 + 5 \times 3$$
$$= 0$$
$$S_6 = \frac{6}{2}(-30 + 5 \times 3)$$
$$= -45$$

17

$$5(2a + 9d) = 120$$
$$2a + 9d = 24 \ldots (1)$$
$$10(2a + 19d) = 840$$
$$2a + 19d = 84 \ldots (2)$$

Equation (2) − Equation (1)

$$10d = 60$$
$$d = 6$$
$$2a + 9 \times 6 = 24$$
$$a = -15$$
$$S_{30} = \frac{30}{2}(-30 + 29 \times 6)$$
$$= 2160$$

18

$$a + 5d = 16 \ldots (1)$$
$$a + 11d = 28 \ldots (2)$$

Equation (2) − Equation (1)

$$6d = 12$$
$$d = 2$$
$$a + 10 = 16$$
$$a = 6$$
$$S_{14} = \frac{14}{2}(12 + 13 \times 2)$$
$$= 266$$

19 a $a + 2d = 6.5 \ldots (1)$

$4(2a + 7d) = 67$

$a + 3.5d = \frac{67}{8} = 8.375 \ldots (2)$

Equation (2) − Equation (1)

$1.5d = 1.875$

$d = 1.25$

$a + 1.25 \times 2 = 6.5$

$a = 4$

$t_n = 4 + 1.25(n - 1)$

$= 2.75 + 1.25n$

$= \frac{5}{4}n + \frac{11}{4}$

b $a + 5d = 17 \ldots (1)$

$5(2a + 9d = 155$

$a + 4.5d = 1505 \ldots (2)$

Equation (1) − Equation (2)

$0.5d = 1.5$

$d = 3$

$a + 5 \times 3 = 17$

$a = 2$

$t_n = 2 + 3(n - 1)$

$= 3n - 1$

20 a $t_{n+1} - t_n = b(n + 1) - bn$

$= b$

b $S_n = \frac{n}{2}(2b + (n - 1)b)$

$= \frac{n}{2}(2b + nb - b)$

$= \frac{n}{2}(nb + b)$

This can be factorised to $\frac{nb(n + 1)}{2}$.

21 $a = 10,\ d = -5$

$t_5 = 10 + 4 \times -5$

$= -10$

$S_{25} = \frac{25}{2}(20 + 24 \times -5)$

$= -1250$

22 $S_{20} = 10(2a + 19d)$

$= 25a$

$20a + 190d = 25a$

$190d = 5a$

$a = 38d$

$S_{30} = 15(76d + 29d)$

$= 1575d$

23 a $S_{n-1} = 17(n - 1) - 3(n - 1)^2$

$= 17n - 17 - 3(n^2 - 2n + 1)$

$= 17n - 17 - 3n^2 + 6n - 3$

$= 23n - 3n^2 - 20$

b $t_n = S_n - S_{n-1}$

$= 17n - 3n^2 - 23n + 3n^2 + 20$

$= 20 - 6n$

c $t_{n+1} - t_n = 20 - 6(n + 1) - (20 - 6n)$

$= 20 - 6n - 6 - 20 + 6n$

$= -6$

The sequence has a constant difference of −6 and so is arithmetic.

$a = t_1$

$= 20 - 6 \times 1 = 14$

$a = 14$

24 Let the terms be $a,\ a+d,\ a+2d$.

$$\text{Sum} = 3a + 3d = 36$$
$$a + d = 12$$
$$\text{Product} = a(a+d)(a+2d) = 1428$$

Substitute $d = 12 - a$.

$$a(a + 12 - a)(a + 24 - 2a) = 1428$$
$$12a(24 - a) = 1428$$
$$a(24 - a) = 119$$
$$24a - a^2 = 119$$
$$a^2 - 24a + 119 = 0$$
$$(a - 7)(a - 17) = 0$$
$$a = 7 \text{ or } a = 17$$
$$\therefore d = 12 - 7 = 5$$
$$\text{or } d = 12 - 17 = -5$$

The three terms are either 7, 12, 17 or 17, 12, 7.

Note: in cases like this, it is sometimes easier to call the terms $a-d,\ a,\ a+d$.

25 The middle terms will be t_n and t_{n+1}.

$$t_n = a + (n-1)d$$
$$t_{n+1} = a + nd$$
$$t_n + t_{n+1} = 2a + (2n-1)d$$
$$n(t_n + t_{n+1}) = n(2a + (2n-1)d)$$
$$S_{2n} = \frac{2n}{2}(2a + (2n-1)d)$$
$$= n(2a + (2n-1)d)$$
$$= n(t_n + t_{n+1})$$

26 There are 60 numbers divisible by 2.
$S_{60} = 30(4 + 59 \times 2) = 3660$
There are 40 numbers divisible by 3.
$S_{40} = 20(6 + 39 \times 3) = 2460$
There are 20 numbers divisible by 6
$S_{60} = 10(12 + 19 \times 6) = 1260$
The sum of the numbers divisible by 2 or 3 = 3660 + 2460 − 1260 = 4860

27 Let the numbers be $a-d, a, a+d, a+2d$.
The sum is $4a + 2d = 100$ which simplifies to $2a + d = 50$.
One solution is $a = 25$ and $d = 0$.
The others are (24, 2), (23, 4), . . . (17, 16)
The sequence for the first solution is 25,25,25,25.
One other sequence is 22,24,26,28.
There are 9 sequences in total.

28 Let the angles be $a-d, a$ and $a+d$.
Then $3a = 180$. Hence $a = 60$. The angles are, $60 - d, 60$ and $60 + d$.
There are 60 such triangles: Listing:
(1, 60, 119), (2, 60, 118), . . . , (60, 60, 60)

Solutions to Exercise 15D

1 $t_n = ar^{n-1}$

a $t_1 = 3 \times 2^{1-1} = 3$

$t_2 = 3 \times 2^{2-1} = 6$

$t_3 = 3 \times 2^{3-1} = 12$

$t_4 = 3 \times 2^{4-1} = 24$

b $t_1 = 3 \times -2^{1-1} = 3$

$t_2 = 3 \times -2^{2-1} = -6$

$t_3 = 3 \times -2^{3-1} = 12$

$t_4 = 3 \times -2^{4-1} = -24$

c $t_1 = 10\,000 \times 0.1^{1-1} = 10\,000$

$t_2 = 10\,000 \times 0.1^{2-1} = 1000$

$t_3 = 10\,000 \times 0.1^{3-1} = 100$

$t_4 = 10\,000 \times 0.1^{4-1} = 10$

d $t_1 = 3 \times 3^{1-1} = 3$

$t_2 = 3 \times 3^{2-1} = 9$

$t_3 = 3 \times 3^{3-1} = 27$

$t_4 = 3 \times 3^{4-1} = 81$

2 a $a = \frac{15}{7}$

$r = \frac{1}{3}$

$t_6 = \frac{15}{7} \times \left(\frac{1}{3}\right)^5 = \frac{5}{567}$

b $a = 1$

$r = -\frac{1}{4}$

$t_5 = 1 \times \left(-\frac{1}{4}\right)^4 = \frac{1}{256}$

c $a = \sqrt{2}$

$r = \sqrt{2}$

$t_{10} = \sqrt{2} \times (\sqrt{2})^9 = 32$

d $a = a^x$

$r = a$

$t_6 = a^x \times a^5 = a^{x+5}$

3 a $a = 3$

$r = \frac{2}{3}$

$t_n = 3 \times \left(\frac{2}{3}\right)^{n-1}$

b $a = 2$

$r = \frac{-4}{2} = -2$

$t_n = 2 \times (-2)^{n-1}$

c $a = 2$

$r = \frac{2\sqrt{5}}{2} = \sqrt{5}$

$t_n = 2 \times (\sqrt{5})^{n-1}$

4 a $a = 2$ and $t_6 = 486$

Let r be the common ratio

$\therefore 2 \times r^5 = 486$

$\therefore r^5 = 243$

$\therefore r = 3$

b $a = 25$ and $t_5 = \frac{16}{25}$

Let r be the common ratio

$\therefore 25 \times r^4 = \frac{16}{25}$

$\therefore r^4 = \frac{16}{625}$

$\therefore r = \pm\frac{2}{5}$

5 $\frac{1}{4}2^{n-1} = 64$

$2^{n-1} = 64 \times 4$

$= 2^8$

$n = 9$

Thus t_9, the ninth term.

6 a $a = 2,\ r = 3$

$2 \times 3^{n-1} = 486$

$3^{n-1} = 243$

$= 3^5$

$n = 6$

b $a = 5,\ r = 2$

$5 \times 2^{n-1} = 1280$

$2^{n-1} = 256$

$= 2^8$

$n = 9$

c $a = 768,\ r = \frac{1}{2}$

$768 \times \left(\frac{1}{2}\right)^{n-1} = 3$

$\frac{1}{2^{n-1}} = \frac{3}{768}$

$= \frac{1}{256} = \frac{1}{2^8}$

$n = 9$

d $a = \frac{8}{9},\ r = \frac{3}{2}$

$\frac{8}{9} \times \frac{3^{n-1}}{2^{n-1}} = \frac{27}{4}$

$\frac{3^{n-1}}{2^{n-1}} = \frac{27}{4} \times \frac{9}{8}$

$= \frac{3^5}{2^5}$

$n = 6$

e $a = -\frac{4}{3},\ r = -\frac{1}{2}$

$-\frac{4}{3} \times \left(-\frac{1}{2}\right)^{n-1} = \frac{1}{96}$

$\left(-\frac{1}{2}\right)^{n-1} = \frac{1}{96} \times -\frac{3}{4}$

$= -\frac{1}{32 \times 4}$

$= -\frac{1}{2^7} = \left(-\frac{1}{2}\right)^7$

$n = 8$

7 $ar^{14} = 54$

$ar^{11} = 2$

$r^3 = \frac{54}{2} = 27$

$r = 3$

$a \times 3^{11} = 2$

$a = \frac{2}{3^{11}}$

$t_7 = \frac{2}{3^{11}} \times 3^6$

$= \frac{2}{3^5}$

8 $ar^1 = \dfrac{1}{2\sqrt{2}}$

$ar^3 = \sqrt{2}$

$r^2 = \sqrt{2} \div \dfrac{1}{2\sqrt{2}}$

$= 4$

$r = 2$

$a \times 2 = \dfrac{1}{2\sqrt{2}}$

$a = \dfrac{1}{4\sqrt{2}}$

$t_8 = \dfrac{1}{4\sqrt{2}} \times 2^7$

$= \dfrac{32}{\sqrt{2}}$

Rationalise the denominator:

$t_8 = \dfrac{32}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$

$= \dfrac{32\sqrt{2}}{2} = 16\sqrt{2}$

9 **a** $ar^5 = 768$

$ar^2 = 96$

$r^3 = \dfrac{768}{96} = 8$

$r = 2$

$a \times 2^2 = 96$

$a = 24$ fish

b $24 \times 2^9 = 12\,888$ fish

10 **a** At the end of 7 days, it will have increased 7 times.
$10 \times 3^7 = 21870$ m^2

b $10 \times 3^n \geq 200\,000$

$3^n \geq 20\,000$

$n \log_{10} 3 \geq \log_{10} 20\,000$

$n \geq 9.014\ldots$

It will cover the lake early in the tenth day.

11 $r = \dfrac{3}{4}$.

First bounce: $\dfrac{3}{2}$ m

Second bounce: $\dfrac{9}{8}$ m

Third bounce: $\dfrac{27}{32}$ m

Fourth bounce: $\dfrac{81}{128}$ m

Fifth bounce: $\dfrac{243}{512}$ m

12 **a** At the end of 10 years, it will have increased 10 times.
$2500 \times 1.08^{10} = \$5397.31$

b $2500 \times 1.08^n \geq 100\,000$

$1.08^n \geq \dfrac{100\,000}{2500} = 40$

$n \log_{10} 1.08 \geq \log_{10} 40$

$n \geq 47.93\ldots$

It will take 48 years until the value exceeds \$100 000. Alternatively, use the solve command of a CAS calculator to solve $2500 \times 1.08^n \geq 100\,000$.

This gives $n > 47.93\ldots$ directly.

13 **a** $120 \times 0.9^7 \approx 57.4$ km

b
$$120 \times 0.9^{n-1} = 30.5$$
$$0.9^{n-1} = \frac{30.5}{120}$$
$$= 0.251\ldots$$
$$(n-1)\log_{10} 0.9 = \log_{10} 0.251\ldots$$
$$n - 1 = 13.0007\ldots$$
$$n = 14$$
The 14th day.

14 $a = 1$ and $r = 2$
$t_{30} = 2^{29} = 5\,368\,709.12$
She would receive \$ 5 368 709.12

15 **a** At the end of 10 years:
$$\text{value} = 5000 \times 1.06^6$$
$$= \$7092.60$$

b
$$1.06^n \geq 2$$
$$n \log_{10} 1.06 \geq \log_{10} 2$$
$$n \geq 11.89\ldots$$
In the 12th year.

16
$$A \times 1.085^{12} = 8000$$
$$A \times 2.6616\ldots = 8000$$
$$A = \$3005.61$$

17 Let the rate be r.
$$r^{10} = 3$$
$$r = 3^{0.1} = 0.11612\ldots$$
Approximately 11.6% per annum.

18
$$a = 4,\ r = 2$$
$$4 \times 2^{n-1} > 2000$$
$$2^{n-1} > 500$$
$$2^9 = 512$$
The tenth term, which is
$t_{10} = 4 \times 2^9 = 2048$

19
$$a = 3,\ r = 3$$
$$3 \times 3^{n-1} > 500$$
$$3^n > 500$$
$$3^5 = 243 \text{ and } 3^6 = 729$$
The sixth term, which is $t_6 = 729$

20 Solve for n:
$$40\,960 \times \left(\frac{1}{2}\right)^{n-1} = 40 \times 2^{n-1}$$
$$\frac{40\,960}{40} = 2^{n-1} \times 2^{n-1}$$
$$1024 = 2^{2n-2} = 2^{10}$$
$$2n - 2 = 10$$
$$n = 6$$
But $n = 1$ corresponds to the initial numbers present, so they are equal after 5 weeks.

21 **a** $\sqrt{5 \times 720} = \sqrt{3600} = 60$

b $\sqrt{1 \times 6.25} = \sqrt{6.25} = 2.5$

c $\sqrt{\frac{1}{\sqrt{3}} \times \sqrt{3}} = \sqrt{1} = 1$

d
$$\sqrt{x^2y^3 \times x^6y^{11}} = \sqrt{x^8y^{14}}$$
$$= x^4y^7$$

22

$$r = \frac{t_7}{t_4} = \frac{t_{16}}{t_7}$$

$$\frac{a + 6d}{a + 3d} = \frac{a + 15d}{a + 6d}$$

$$(a + 6d)^2 = (a + 15d)(a + 3d)$$

$$a^2 + 12ad + 36d^2 = a^2 + 18ad + 45d^2$$

$$9d^2 + 6ad = 0$$

$$3d(3d + 2a) = 0$$

$$3d + 2a = 0 \text{ (see below)}$$

$$d = -\frac{2}{3}a$$

$$r = \frac{a + 6d}{a + 3d}$$

$$= \frac{a - 4a}{a - 2a}$$

$$= \frac{-3a}{-a} = 3$$

Note: $d = 0$ gives the trivial case $r = \frac{a}{a} = 1$.
(All the terms are the same.)

23 $a^{n-1} + a^n = a^{n+1}$

$$\therefore a^{n-1}(1 + a - a^2) = 0$$

$$\therefore a = \frac{1 \pm \sqrt{5}}{2} \text{ or } a = 0$$

24 **a** When the first 300 mL is withdrawn there is 700 mL of ethanol left.
When the second 300 mL withdrawn there is $0.7^2 \times 1000$ mL of ethanol left
After 5 such withdrawals there is $0.7^5 \times 1000 \approx 168.07$ mL left.

b Solve the inequality $1000 \times 0.7^n < 1$ for n an integer t find n =20.

25 **a** The perimeter of the rectangle is $2a + 2b$. Each side of the corresponding square will be $\frac{a+b}{2}$, the arithmetic mean of a and b.

b The area of the rectangle is ab. The side length of the corresponding square is $\sqrt{ab}$, the geometric mean of a and b.

Solutions to Exercise 15E

1 $S_n = \dfrac{a(r^n - 1)}{r - 1}$

a $a = 5$

$r = \dfrac{10}{5} = 2$

$S_{10} = \dfrac{5(2^{10} - 1)}{2 - 1}$

$= 5115$

b $a = 1$

$r = \dfrac{-3}{1} = -3$

$S_6 = \dfrac{1(-3^6 - 1)}{-3 - 1}$

$= -182$

c $a = -\dfrac{4}{3}$

$r = \dfrac{2}{3} \div -\dfrac{4}{3} = -\dfrac{1}{2}$

$S_9 = \dfrac{-\dfrac{4}{3}\left(\left(-\dfrac{1}{2}\right)^9 - 1\right)}{-\dfrac{1}{2} - 1}$

$= -\dfrac{57}{64}$

2 a $a = 2$

$r = \dfrac{-6}{2} = -3$

$t_n = 1458 = 2 \times -3^{n-1}$

$-3^{n-1} = 729$

$n = 7$

$S_7 = \dfrac{2 \times (-3^7 - 1)}{-3 - 1}$

$= 1094$

b $a = -4$

$r = \dfrac{8}{-4} = -2$

$t_n = -1024 = -4 \times -2^{n-1}$

$-2^{n-1} = 256$

$n = 9$

$S_9 = \dfrac{-4 \times (-2^9 - 1)}{-2 - 1}$

$= -684$

c $a = 6250$

$r = \dfrac{1250}{6250} = 0.2$

$t_n = 2 = 6250 \times (0.2)^{n-1}$

$(0.2)^{n-1} = \dfrac{2}{6250} = \dfrac{1}{3125}$

$n = 6$

$S_6 = \dfrac{6250 \times ((0.2)^6 - 1)}{0.2 - 1}$

$= 7812$

3 $a = 3$ and $r = 2$

$\therefore S_n = \dfrac{3(2^n - 1)}{2 - 1}$

If $S_n = 3069$ then

$3(2^n - 1) = 3069$

$2^n - 1 = 1023$

$2^n = 1024$

$n = 10$

4 $a = 24$ and $r = -\frac{1}{2}$

$\therefore S_n = \dfrac{24(1 + (\frac{1}{2})^n)}{1 + \frac{1}{2}}$

$S_n = 16(1 - (\frac{1}{2})^n)$

If $S_n = \dfrac{129}{8}$ then

$16(1 + (\frac{1}{2})^n) = \dfrac{129}{8}$

$1 + (\frac{1}{2})^n = \dfrac{129}{128}$

$(\frac{1}{2})^n = \dfrac{1}{128}$

$n = 7$

5 $a = 600, \ r = 1.1$

a $t_7 = 600 \times 1.1^6$

$= 1062.9366$

About 1062.9 mL

b $S_7 = \dfrac{600 \times (1.1^7 - 1)}{1.1 - 1}$

$= 5692.3026$

About 5692.3 mL

c 11 days

6 $a = 20, \ r = \dfrac{25}{20} = 2.5$

a $t_5 = 20 \times 1.25^4$

$= 48.828125$

49 minutes (to the nearest minute)

b $S_5 = \dfrac{20 \times (1.25^5 - 1)}{1.25 - 1}$

$= 164.140625$

164 minutes, or 2 hours and 44 minutes

c $S_n > 15 \times 60 = 900$

$\dfrac{20 \times (1.25^n - 1)}{0.25} > 900$

$1.25^n - 1 > 900 \times \dfrac{0.25}{20}$

$= 11.25$

$1.25^n > 12.25$

$n \log_{10} 1.25 > \log_{10} 12.25$

$n > 11.228$

$12 - 7 = 5$, so Friday.

7 $a = 15, \ r = \dfrac{2}{3}$

$$S_{10} = \frac{15 \times \left(1 - \left(\frac{2}{3}\right)^{10}\right)}{1 - \frac{2}{3}}$$

$$= 3 \times 15 \times \frac{3^{10} - 2^{10}}{3^{10}}$$

$$= 5 \times \frac{3^{10} - 2^{10}}{3^8}$$

$$= \frac{5 \times 58\,025}{6561}$$

$$= \frac{290\,125}{6561}$$

The bounces will all be doubled (up and down) except for the first (down only).

$$\text{Distance} = 2 \times \frac{290\,125}{6561} - 15$$

$$= \frac{481\,835}{6561}$$

$$= 73\frac{2882}{6561} \text{ m}$$

8 $a = \$15\,000, \ r = 1.05$

a $t_5 = 15\,000 \times 1.05^4$

$= 18\,232.593\ldots$

\$18 232.59

b $S_5 = \frac{15\,000 \times (1.05^5 - 1)}{1.05 - 1}$

$= 82\,844.4686$

\$82 884.47

9 Andrew: Interest = $1000 \times 0.20 \times 10$

= \$2000

His investment is worth

\$1000 + \$2000 = \$3000.

Bianca's investment is worth

$1000 \times 1.125^{10} = \$3247.32$

Bianca's investment is worth more.

10 a

$$ar^2 = 20$$

$$ar^5 = 160$$

$$r^3 = \frac{160}{20} = 8$$

$$r = 2$$

$$a \times 2^2 = 20$$

$$a = 5$$

$$S_5 = \frac{5 \times (2^5 - 1)}{2 - 1}$$

$$= 155$$

b

$$ar^2 = \sqrt{2}$$

$$ar^7 = 8$$

$$r^5 = \frac{8}{\sqrt{2}}$$

$$= \frac{\sqrt{64}}{\sqrt{2}}$$

$$= \sqrt{32} = (\sqrt{2})^5$$

$$r = \sqrt{2}$$

$$a \times (\sqrt{2})^2 = \sqrt{2}$$

$$a = \frac{1}{\sqrt{2}}$$

$$S_8 = \frac{\frac{1}{\sqrt{2}} \times ((\sqrt{2})^8 - 1)}{\sqrt{2} - 1}$$

$$= \frac{\frac{1}{\sqrt{2}} \times 15}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1}$$

$$= \frac{\frac{15}{\sqrt{2}} \times (\sqrt{2} + 1)}{2 - 1}$$

$$= 15 + \frac{15\sqrt{2}}{2}$$

11 $a = 1,\ r = 2$

a

$$S_n = 255$$

$$\frac{1 \times (2^n - 1)}{2 - 1} = 255$$

$$2^n - 1 = 255$$

$$2^n = 256$$

$$n = 8$$

b
$$
\begin{aligned}
S_n &> 1\,000\,000 \\
\frac{1 \times (2^n - 1)}{2 - 1} &> 1\,000\,000 \\
2^n - 1 &> 1\,000\,000 \\
2^n &> 1\,000\,001 \\
n \log_{10} 2 &> \log_{10} 1\,000\,001 \\
n &> 19.931\ldots
\end{aligned}
$$
$\{n : n > 19\}$ or $\{n : n \geq 20\}$, since n is a positive integer.

12 $a = 1,\ r = -x^2$
Note that there are $(m + 1)$ terms.
$$
\begin{aligned}
S_{m+1} &= \frac{1 \times (-x^2)^{m+1} - 1}{-x^2 - 1} \\
&= \frac{-x^{2(m+1)} - 1}{-x^2 - 1} \\
&= \frac{x^{2m+2} + 1}{x^2 + 1}
\end{aligned}
$$

13 **a** The thickness of each piece is 0.05 mm.
There are $\cdots 2^{40}$ pieces of paper of this thickness.
The thickness is $2^{40} \times 0.05 \approx$ 54976 km

b Solve the inequality $0.05 \times 2^n \geq 384400 \times 10^6$ for n an integer to find $n = 43$.

14 Option 1: \$52 million;
Option 2: \$45 040 000 million

Solutions to Exercise 15F

1 $S_\infty = \dfrac{a}{1-r}$

a $a = 1$

$r = \dfrac{1}{5} \div 1 = \dfrac{1}{5}$

$S_\infty = \dfrac{1}{1-\dfrac{1}{5}}$

$= \dfrac{5}{4}$

b $a = 1$

$r = -\dfrac{2}{3} \div 1 = -\dfrac{2}{3}$

$S_\infty = \dfrac{1}{1--\dfrac{2}{3}}$

$= \dfrac{3}{5}$

2 Each side, and hence each perimeter, will be half the larger side.

$r = \dfrac{1}{2},\ a = p$

$\text{Perimeter of } n\text{th triangle} = p \times \left(\dfrac{1}{2}\right)^{n-1}$

$= \dfrac{p}{2^{n-1}}$

$S_\infty = \dfrac{p}{1-\dfrac{1}{2}}$

$= 2p$

$\text{Area} = \dfrac{p^2\sqrt{3}}{9 \times 4^n}$

$\text{Sum of the areas} = \dfrac{p^2\sqrt{3}}{27}$

3 $a = 200,\ r = 0.94$

$S_\infty = \dfrac{200}{1-0.94}$

$= 3333\dfrac{1}{3}$ m

4 $a = 450,\ r = 0.65$

$S_\infty = \dfrac{450}{1-0.65}$

≈ 1285.7

Yes, it will kill him.

5 $a = 3,\ r = 0.5$

$S_\infty = \dfrac{3}{1-0.5} = 6$

He can only make the joumey if he walks for an infinite time (which isn't very likely).

6 $a = 2,\ r = \dfrac{3}{4}$

$S_\infty = \dfrac{2}{1-0.75} = 8$

The frog will approach a limit of 8 m.

7 $a = \dfrac{1}{3},\ r = \dfrac{1}{3}$

$S_\infty = \dfrac{\dfrac{1}{3}}{1-\dfrac{1}{3}}$

$= \dfrac{1}{2}$ or 50%

8 $r = 70\% = 0.7$

$S_\infty = \dfrac{a}{1-0.7} = 40$

$a = 0.3 \times 40$

$= 12$ m

9 Note: all distances will be double (up and down) except the first (down only). Use $a = 30,\ r = \dfrac{2}{3}$ and subtract 15 m from the answer.

$$S_\infty = \frac{30}{1-\frac{2}{3}} = 90$$

Distance = 90 − 15 = 75 m

10 **a** $a = 0.4,\ r = 0.1$

$$S_\infty = \frac{0.4}{1-0.1} = \frac{4}{9}$$

b $a = 0.03,\ r = 0.1$

$$S_\infty = \frac{0.03}{1-0.1} = \frac{3}{90} = \frac{1}{30}$$

c $a = 0.3,\ r = 0.1$

$$S_\infty = \frac{0.3}{1-0.1} = \frac{3}{9} = \frac{1}{3}$$

Decimal $= 10\frac{1}{3} = \frac{31}{3}$

d $a = 0.035,\ r = 0.01$

$$S_\infty = \frac{0.035}{1-0.01} = \frac{35}{990} = \frac{7}{198}$$

e $a = 0.9,\ r = 0.1$

$$S_\infty = \frac{0.9}{1-0.1} = \frac{9}{9} = 1$$

f $a = 0.1,\ r = 0.1$

$$S_\infty = \frac{0.1}{1-0.1} = \frac{1}{9}$$

Decimal $= 4\frac{1}{9} = \frac{37}{9}$

11 $S_4 = \dfrac{a(1-r^4)}{1-r} = 30$

$S_\infty = \dfrac{a}{1-r} = 32$

$a = 32(1-r)$

Substitute for a:

$$\frac{32(1-r)(1-r^4)}{1-r} = 30$$

$$32(1-r^4) = 30$$

$$1 - r^4 = \frac{30}{32}$$

$$r^4 = 1 - \frac{30}{32} = \frac{2}{32} = \frac{1}{16}$$

$$r = \frac{1}{2} \text{ or } r = -\frac{1}{2}$$

If $r = \frac{1}{2}$: $a = 32\left(1 - \frac{1}{2}\right) = 16$

If $r = -\frac{1}{2}$: $a = 32\left(1 - \left(-\frac{1}{2}\right)\right) = 48$

The first two terms are 16 and 8, or 48 and −24

12 $S_\infty = \dfrac{a}{1+\frac{1}{4}}$

$= \dfrac{4a}{5} = 8$

$a = 10$

$t_3 = 10 \times \left(-\dfrac{1}{4}\right)^2$

$= \dfrac{5}{8}$

13 $\dfrac{5}{1-r} = 15$

$5 = 15(1-r)$

$1 - r = \dfrac{1}{3}$

$r = 1 - \dfrac{1}{3}$

$= \dfrac{2}{3}$

14 $\dfrac{2}{1-r} = x$

Solve for r

$\dfrac{2}{x} = 1 - r$

$r = 1 - \dfrac{2}{x}$

Since $x > 2$, $\dfrac{2}{x} < 1$ and so $r = 1 - \dfrac{2}{x} < 1$

Solutions to short-answer questions (technology-free)

1 **a** $t_1 = 3$

$t_2 = 3 - 4 = -1$

$t_3 = -1 - 4 = -5$

$t_4 = -5 - 4 = -9$

$t_5 = -9 - 4 = -13$

$t_6 = -13 - 4 = -17$

b $t_1 = 5$

$t_2 = 2 \times 5 + 2 = 12$

$t_3 = 2 \times 12 + 2 = 26$

$t_4 = 2 \times 26 + 2 = 54$

$t_5 = 2 \times 54 + 2 = 110$

$t_6 = 2 \times 110 + 2 = 222$

2 **a** $t_1 = 2 \times 1 = 2$

$t_2 = 2 \times 2 = 4$

$t_3 = 2 \times 3 = 6$

$t_4 = 2 \times 4 = 8$

$t_5 = 2 \times 5 = 10$

$t_6 = 2 \times 6 = 12$

b $t_1 = -3 \times 1 + 2 = -1$

$t_2 = -3 \times 2 + 2 = -4$

$t_3 = -3 \times 3 + 2 = -7$

$t_4 = -3 \times 4 + 2 = -10$

$t_5 = -3 \times 5 + 2 = -13$

$t_6 = -3 \times 6 + 2 = -16$

3 **a** End of first year:
$\$5000 \times 1.05 = \$5250$
Start of second year:
$\$5250 + \$500 = \$5750$
End of second year:
$\$5750 \times 1.05 = \$6037.50$

b $t_n = 1.05(t_{n-1} + 500),\ t_1 = 5250$

4 $a + 3d = 19 \ldots (1)$

$a + 6d = 43 \ldots (2)$

Equation (2) – Equation (1)

$3d = 24$

$d = 8$

$a + 3 \times 8 = 19$

$a = -5$

$t_{20} = -5 + 19 \times 8$

$= 147$

5 $a + 4d = 0.35 \ldots (1)$

$a + 8d = 0.15 \ldots (2)$

Equation (2) – Equation (1)

$4d = -0.2$

$d = -0.05$

$a + 4 \times -0.05 = 0.35$

$a = 0.35 + 0.2$

$= 0.55$

$t_{14} = 0.55 + 13 \times -0.55$

$= -0.1$

6 $a + 5d = -24 \ldots (1)$

$a + 13d = 6 \ldots (2)$

Equation (2) − Equation (1)

$8d = 30$

$d = 3.75$

$a + 5 \times 3.75 = -24$

$a = -24 - 18.75$

$= -42.75$

$S_{10} = 5 \times (-85.5 + 9 \times 3.75)$

$= -258.75$

7 $a = -5,\ d = 7$

$S_n = \frac{n}{2}(-10 + 7(n-1))$

$= 402$

$n(-10 + 7(n-1)) = 804$

$7n^2 - 10n - 7n = 804$

$7n^2 - 17n - 804 = 0$

$(7n + 67)(n - 12) = 0$

$n = 12$ (since $n > 0$)

$\{n : S_n = 402\} = \{n : n = 12\}$

8 $ar^5 = 9$

$ar^9 = 729$

$r^4 = 81$

$r = 3$ or $r = -3$

$r = 3 : a \times 3^5 = 9$

$a = \frac{9}{243} = \frac{1}{27}$

$t_4 = \frac{1}{27} \times 3^3 = 1$

$r = -3 : a \times (-3)^5 = 9$

$a = -\frac{1}{27}$

$t_4 = -\frac{1}{27} \times (-3)^3 = 1$

So for either case, $t_4 = 1$

9 $a = 1000$

$r = 1.035$

$t_n = ar^n$

$= 1000 \times 1.035^n$

10 $9r^2 = 4$

$r^2 = \frac{4}{9}$

$r = \pm\frac{2}{3}$

$t_2 = ar = \pm 6$

$t_4 = ar^3 = \pm\frac{8}{3}$

Terms $= 6, \frac{8}{3}$ or $-6, -\frac{8}{3}$

11 $a + ar + ar^2 = 24$

$ar^3 + ar^4 + ar^5 = 24$

$r^3(a + ar + ar^2) = 24$

$r^3 = 1$

$r = 1$

All terms will be the same: $t_n = \frac{24}{3} = 8$

$S_{12} = 12 \times 8 = 96$

12 $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$$S_8 = \frac{6 \times (-3^8 - 1)}{-3 - 1}$$

$$= -9840$$

13 $a = 1,\ r = -\dfrac{1}{3}$

$$S_\infty = \frac{1}{1 - -\frac{1}{3}}$$

$$= \frac{3}{4}$$

14

$$\frac{x + 4}{x} = \frac{2x + 2}{x + 4}$$

$$(x + 4)^2 = x(2x + 2)$$

$$x^2 + 8x + 16 = 2x^2 + 2x$$

$$2x^2 + 2x - x^2 - 8x - 16 = 0$$

$$x^2 - 6x - 16 = 0$$

$$(x - 8)(x + 2) = 0$$

$$x = 8 \text{ or } x = -2$$

Solutions to multiple-choice questions

1 D $t_1 = 3 \times 1 + 2 = 5$

$t_2 = 3 \times 2 + 2 = 8$

$t_3 = 3 \times 3 + 2 = 11$

2 B $t_2 = 3 + 3 = 6$

$t_3 = 6 + 3 = 9$

$t_4 = 9 + 3 = 12$

3 A $a = 10$

$d = 8 - 10 = -2$

$t_{10} = 10 + (9 \times -2)$

$= -8$

4 A $a = 10,\ d = -2$

$S_{10} = \dfrac{10}{2}(10 + -8)$

$= 10$

5 B $a = 8$

$d = 13 - 8 = 5$

$t_n = 8 + 5(n - 1) = 58$

$5(n - 1) = 50$

$n - 1 = 10$

$n = 11$

6 D $a = 12$

$r = \dfrac{8}{12} = \dfrac{2}{3}$

$t_6 = 12 \times \left(\dfrac{2}{3}\right)^5$

$= \dfrac{128}{81}$

7 E $a = 8$

$r = \dfrac{4}{8} = \dfrac{1}{2}$

$S_6 = \dfrac{8 \times \left(1 - \left(\frac{1}{2}\right)^6\right)}{1 - \frac{1}{2}}$

$= 15\dfrac{3}{4}$

8 C $a = 8$

$r = \dfrac{4}{8} = \dfrac{1}{2}$

$S_\infty = \dfrac{8}{1 - \frac{1}{2}}$

$= 16$

9 E Value $= 2000 \times 1.055^6$

$= \$2757.69$

10 D $\dfrac{a}{1 - \frac{1}{3}} = 37.5$

$a = 37.5 \times \dfrac{2}{3}$

$= 25$

Solutions to extended-response questions

1 **a** 0.8, 1.5, 2.2, . . .

b $d = 0.7$ and so the sequence is conjectured to be arithmetic.

c
$$\begin{aligned} t_n &= 0.8 + (n-1) \times 0.7 \\ \therefore t_{12} &= 0.8 + (12-1) \times 0.7 \\ &= 8.5 \end{aligned}$$
The length of moulding in the kit size 12 is 8.5 metres.

2 **a** $d = 25$ and so the sequence is arithmetic.

b
$$\begin{aligned} t_n &= a + (n-1)d \\ &= 50 + (n-1) \times 25 \\ &= 50 + 25n - 25 \\ &= 25n + 25 \end{aligned}$$

c
$$\begin{aligned} t_{25} &= 25 \times 25 + 25 \\ &= 650 \end{aligned}$$
There are 650 seeds in the 25th size packet.

3 The distances $5, 5-d, \ 5-2d, \ldots, 5-6d$ form an arithmetic sequence of seven terms with common difference $-d$.

$$\begin{aligned} \text{Now } S_n &= \frac{n}{2}(a + \ell) \\ \therefore S_7 &= \frac{7}{2}(5 + 5 - 6d) \\ \text{Since } S_7 = 32 - 3 = 29, \ 29 &= \frac{7}{2}(10 - 6d) \\ \therefore \frac{58}{7} &= 10 - 6d \\ \therefore 6d &= \frac{12}{7} \\ \therefore d &= \frac{2}{7} \end{aligned}$$

The distance of the fifth pole from town A is given by S_5.

$$\begin{aligned} S_5 &= \frac{5}{2}\left(5 + 5 - 4 \times \frac{2}{7}\right) \\ &= \frac{155}{7} \\ &= 22\frac{1}{7} \text{ and } 32 - 22\frac{1}{7} = 9\frac{6}{7} \end{aligned}$$

The fifth pole is $22\frac{1}{7}$ km from town A and $9\frac{6}{7}$ km from town B.

4 a 20, 36, 52, 68, 84, 100, 116, 132, . . .

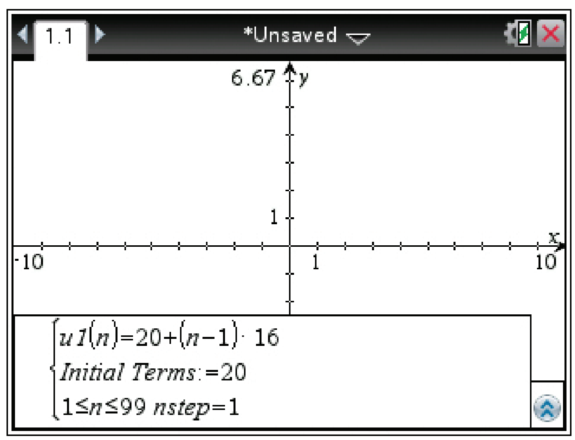
b
$$\begin{aligned} T_n &= a + (n-1)d \\ &= 20 + (n-1) \times 16 \\ &= 20 + 16n - 16 \\ &= 16n + 4 \end{aligned}$$

c
$$\begin{aligned} \text{Let } T_n &= 196 \\ \therefore 16n + 4 &= 196 \\ \therefore 16n &= 192 \\ \therefore n &= 12 \end{aligned}$$

Yes, size 12 will handle 196 lines.

CAS calculator techniques for Question 4

TI: Open a Graphs page. Press **Menu** → **3** : **Graph Entry/Edit** → **6** : **Sequence** → **1** : **Sequence** and input the equation and initial term as shown. Press ENTER then press /T to view the sequence.

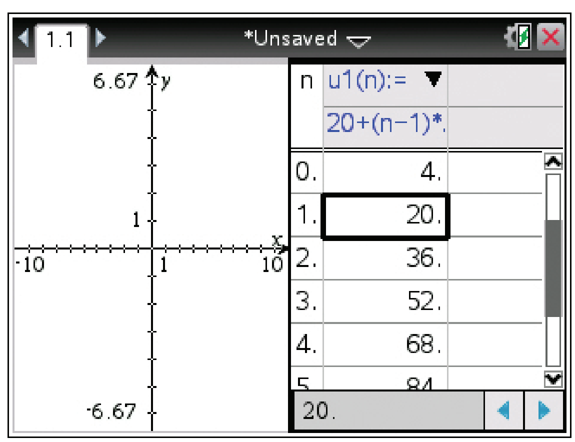


CP: Open the Sequence application.
Input the following:

$a_{n+1} = 20 + (n-1) \times 16$

$a_0 = 20$

Tap # to view the sequence.



5 a $D_n = a + (n - 1)d$

$= 2 + (n - 1) \times 7$

$= 2 + 7n - 7$

$= 7n - 5$

b $D_{n+1} = 191$

$\therefore 7(n + 1) - 5 = 191$

$\therefore 7(n + 1) = 196$

$\therefore n + 1 = 28$

$\therefore n = 27$

The firm made 27 different thicknesses.

6 $t_1 = 4,\ t_2 = 16,\ t_3 = 28 \qquad \therefore d = 12$

$t_{40} = a + (40 - 1)d$

$= 4 + 39 \times 12$

$= 472$

The house will slip 472 mm in the 40th year.

7 $t_1 = 16,\ t_2 = 24,\ t_3 = 32 \qquad \therefore d = 8$

$S_{10} = \frac{10}{2}(2 \times 16 + (10 - 1) \times 8)$

$= 5\,(32 + 72)$

$= 520$

She will have sent 520 cards altogether in 10 years.

8 a $a = 90,\ r = \frac{1}{10},$

$$\therefore S_6 = \frac{90\left(1 - \left(\frac{1}{10}\right)^6\right)}{1 - \frac{1}{10}}$$

$= 99.9999$

After six rinses, Joan will have washed out 99.9999 mg of shampoo.

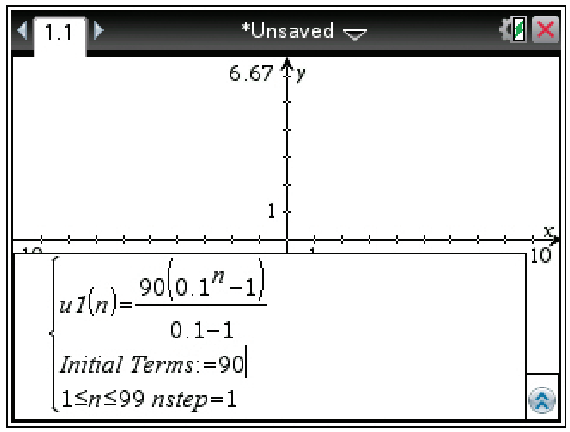
b

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{90}{1-\frac{1}{10}} \\ &= 100 \end{aligned}$$

There were 100 mg present at the beginning.

CAS calculator techniques for Question 8

TI: Open a Graphs page. Press **Menu** → **3** : **Graph Entry/Edit** → **6** : **Sequence** → **1** : **Sequence** and input the equation and initial term as shown. Press ENTER then press /T to view the sequence.

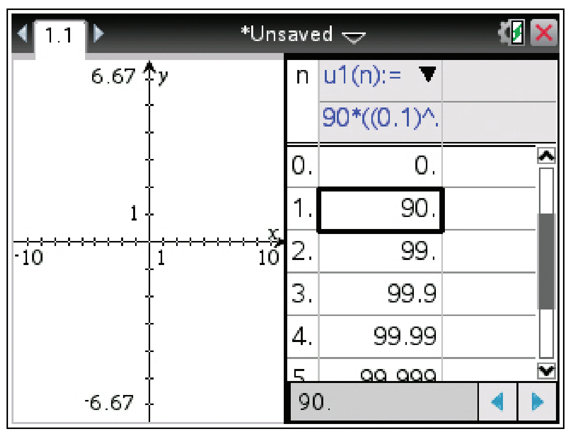


CP: Open the Sequence application. Input the following:

$$\boldsymbol{a_{n+1} = \frac{90(0.1^n - 1)}{0.1 - 1}}$$

$$\boldsymbol{a_0 = 90}$$

Tap # to view the sequence.



9 **a** $t_1 = \frac{1}{3}$, $t_2 = \left(\frac{1}{3}\right)^2$, $t_6 = \left(\frac{1}{3}\right)^6 = \frac{1}{729}$

The water level will rise by $\frac{1}{729}$ metres at the end of the sixth hour.

b $S_6 = \dfrac{\frac{1}{3}\left(1 - \left(\frac{1}{3}\right)^6\right)}{1 - \frac{1}{3}}$

$= \dfrac{364}{729}$

$= 0.499314\ldots$

The total height of the water level after six hours will be 1.499 m, correct to three decimal places.

$S_\infty = \dfrac{\frac{1}{3}}{1 - \frac{1}{3}}$

$= 0.5$

The maximum height the water will reach is 1.5 metres. If the prisoner is able to keep his head above this level, he will not drown.

10 **a** $\dfrac{400}{500} = \dfrac{320}{400} = 0.8$

$a = 500,\ r = 0.8,$

$\therefore t_n = 500(0.8)^{n-1}$

$t_{14} = 500(0.8)^{14-1}$

$= 27.487\,790\ldots$

On the 14th day they were subjected to 27.49 curie hours, correct to two decimal places.

b $S_n = \dfrac{a(1 - r^n)}{1 - r}$

$S_5 = \dfrac{500(1 - 0.8^5)}{1 - 0.8}$

$= 1680.8$

During the first five days, they were subjected to 1680.8 curie hours.

11 **a** $t_1 = \dfrac{2}{3} \times 81$

$t_2 = \left(\dfrac{2}{3}\right)^2 \times 81$

$t_6 = \left(\dfrac{2}{3}\right)^6 \times 81$

$= 7\dfrac{1}{9}$

After the sixth bounce, the ball reaches a height of $7\frac{1}{9}$ metres.

b Total distance $= 81 + \frac{2}{3} \times 81 + \frac{2}{3} \times 81 + \left(\frac{2}{3}\right)^2 \times 81 + \left(\frac{2}{3}\right)^2 \times 81 + \ldots$

$$= 81 + 2\left(\frac{2}{3} \times 81 + \left(\frac{2}{3}\right)^2 \times 81 + \ldots\right)$$

$$= 81 + 2 \times \frac{\frac{2}{3} \times 81}{1 - \frac{2}{3}}$$

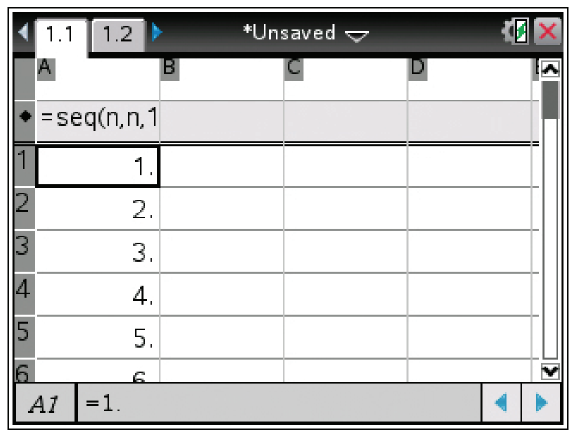
$$= 81 + 324$$

$$= 405$$

The total distance travelled by the ball is 405 metres.

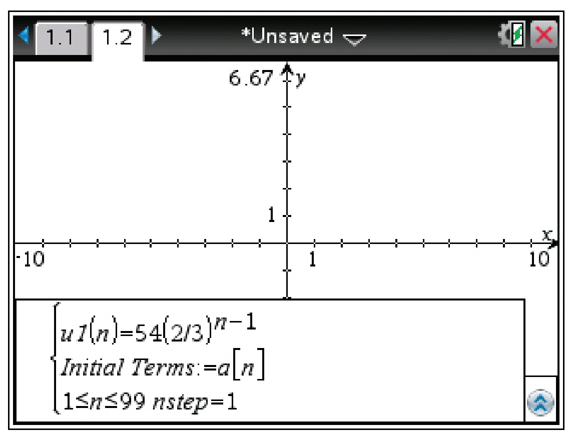
CAS calculator techniques for Question 11

TI: Open a Lists & Spreadsheet application. Type **seq**(**n**, **n**, **1**, **30**, **1**) into the formula cell for column A. This will place the number 1–30 into column A.



Open a Graphs application and input the following sequence.

Navigate back to the Lists & Spread-sheet page. Type **seq**(**ul**(**n**), **n**, **1**, **30**, **1**) into the formula cell for columns B.



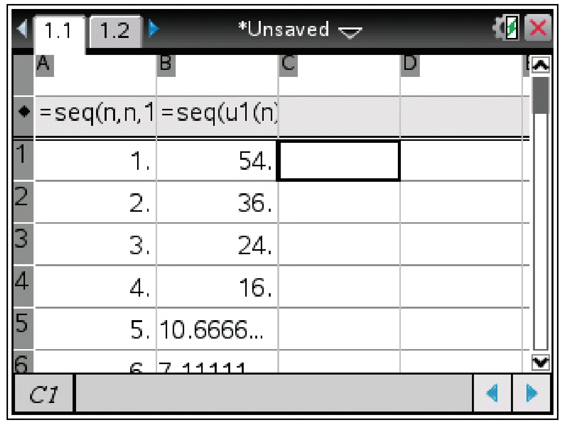
Type **2** × **b**[] into the formula cell for column C.

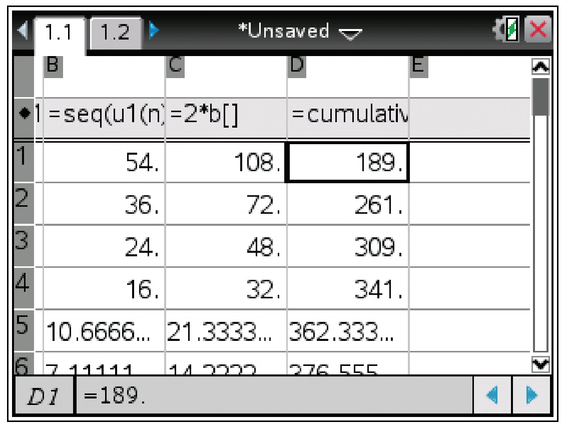
Type **cumulativeSum**(**c**[]) + **81** into the formula cell for column D.

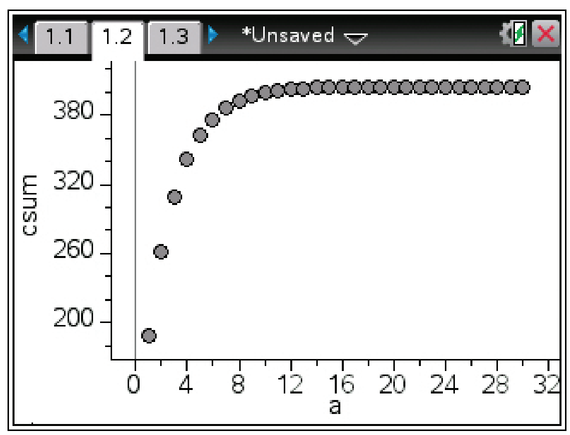
Give column D the name **csum** and column A the name **a**

The graphs of these relations can now be considered. In a Data & Statistics application sketch the graph of **csum** against **a** as shown. This is the total distance travelled against the number of bounces.

The limiting behaviour is demonstrated by this graph.







12 $t_1 = 1 = 2^0$

$t_2 = 2 = 2^1$

$t_3 = 4 = 2^2$

$\therefore t_n = 2^{n-1}$

$S_n = \dfrac{a(1 - r^n)}{1 - r}$ where $a = 1,\ r = 2$

$\therefore S_{64} = \dfrac{1(1 - 2^{64})}{1 - 2}$

$= 2^{64} - 1$

The king had to pay $2^{64} - 1 = 1.845 \times 10^{19}$ grains of rice.

13 a i The amount of cement produced is an arithmetic sequence.

Let C_n be the amount of cement produced (in tonnes) in the nth month.

$C_n = a + (n - 1)d$ where $a = 4000,\ d = 250$

$= 4000 + (n - 1) \times 250$

$= 4000 + 250n - 250$

$\therefore C_n = 250n + 3750$

ii Let S_n be the amount of cement (in tonnes) produced in the first n months.

$S_n = \dfrac{n}{2}(a + 1)$ where $a = 4000,\ l = 250n + 3750$

$= \dfrac{n}{2}(4000 + 250n + 3750)$

$= \dfrac{n}{2}(250n + 7750)$

$= n(125n + 3875)$

$\therefore S_n = 125n(n + 31)$

$= 3875n + 125n^2$

iii When $C_n = 9250$,

$250n + 3750 = 9250$

$\therefore 250n = 5500$

$\therefore n = 22$

The amount of cement produced is 9250 tonnes in the 22nd month.

iv $C_n = 250n + 3750$

$\therefore T = 250m + 3750$

$\therefore m = \dfrac{1}{250}T - 15$

v $S_p = 522\,750$ and $S_p = \frac{p}{2}(a+l)$

$$\therefore 522\,750 = \frac{p}{2}(4000 + 250p + 3750)$$

$$\therefore 1045\,500 = p(250p + 7750)$$

$$\therefore 4182 = p(p+31)$$

$\therefore p^2 + 31p - 4182 = 0$

Using the general quadratic formula,

$$p = \frac{-31 \pm \sqrt{31^2 - 4 \times 1 \times (-4182)}}{2}$$

$$= \frac{-31 \pm 131}{2}$$

$$= -82 \text{ or } 51$$

$$= 51 \text{ as } p > 0$$

b **i** The total amount of cement produced is a geometric series. Total amount of cement produced after n months is given by

$$S_n = \frac{a(r^n - 1)}{r - 1} \text{ where } a = 3000,\ r = 1.08$$

$$= \frac{3000(1.08^n - 1)}{0.08}$$

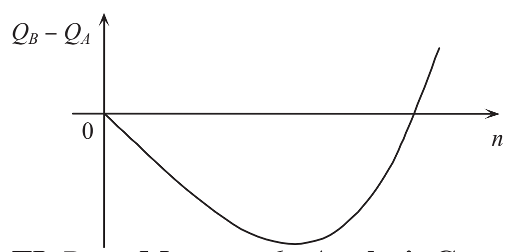
$\therefore S_n = 37500(1.08^n - 1)$

ii $Q_A = 125n(n+31)$ and $Q_B = 37\,500(1.08^n - 1)$

$\therefore Q_B - Q_A = 37\,500(1.08^n - 1) - 125n(n+31)$

Using a CAS calculator,

sketch $\mathbf{f1} = \mathbf{37\,500(1.08^x} - 1) - \mathbf{125x(x + 31)}$



TI: Press **Menu** → **6** : **Analysis Graph** → **1** : **Zero**

CP: Tap **Analysis** → **G - Solve** → **Root** to yield a horizontal axis intercept at (17.28, 0), correct to two decimal places. Hence, the smallest value of n for which $Q_B - Q_A \geq 0$ is 18.

14 **a** Geometric sequence with $a = 1$ and $r = 3$:

Number of white triangles after step n is 3^{n-1}

b Geometric sequence with $a = 1$ and $r = \frac{1}{2}$

Side length of white triangle in diagram n is $\left(\frac{1}{2}\right)^{n-1}$

c Geometric sequence with $a = 1$ and $r = \frac{3}{4}$:

Fraction that is white $= \left(\frac{3}{4}\right)^{n-1}$

d As $n \to \infty$ the fraction that is white approaches 0.

15 **a** Geometric sequence with $a = 1$ and $r = 8$:
Number of white squares after step n is 8^{n-1}

b Geometric sequence with $a = 1$ and $r = \frac{1}{3}$:

Side length of white square in diagram n is $\left(\frac{1}{3}\right)^{n-1}$

c Geometric sequence with $a = 1$ and $r = \frac{8}{9}$:

Fraction that is white $= \left(\frac{8}{9}\right)^{n-1}$

d As $n \to \infty$ the fraction that is white approaches 0.